Timberlane Regional High School **PRECALCULUS ACC** **Summer Work 2019 - 2020**

PreCalculus is the gateway to any future mathematics course and many science courses. Success will be attained through a positive attitude, hard work, and perseverance, especially when topics seem extremely difficult. Time is a constraint that will prohibit us from learning the material required of a quality PreCalculus course unless we spend some time prior to the start of class reviewing topics you have encountered in previous math courses.

The first chapter of the PreCalculus text is titled *Fundamentals*. There are seven sections containing a number of topics ranging from Geometry, Algebra I and II. It is expected these topics are mastered and therefore truly fundamentally basic. Outlined below are four sections of practice questions to complete. For each topic below, work each problem. Take care and show pride in your work. **Be prepared to hand in quality work; work that is neat and organized for the final solution.** Please include multiple attempts (if necessary) to arrive at a correct solution. We want to see perseverance when needed. This assigned practice work is to check if you fully comprehend these topics. Work each problem to the best of your ability. The odd answers have been provided for you to try some practice problems and then be able to check your work in the answers section. If you are unsuccessful, try looking for careless errors or rework the problem from another perspective. Take the time and effort to reference examples from the text, call a friend or look for assistance from the internet (such as Kahn Academy) for suggestions on the topic to assist you in your problem solving endeavors! In other words, don't give up! Each of the four review sections below will be worth 25 points. Math assignments grow in challenge as the practice problems increase. Thus, the first five assigned problems are graded on correctness only for one point each. The remaining five assigned problems will be assessed on two points for perseverance/effort and two points for accurately arriving at the correct solution. Summer work is due Thursday September 5th and Friday September 6th. Summer work will account for 3% of your first quarter average. Late work will be penalized 10 points per day. No late work will be accepted after September 12th.

Topic	*Assignment*	*First five assigned problems... one point each for correctness*	*Second five assigned problems... two points each for effort two points each for correctness*	*Total*
1.1 Sets of Real $\Re$ Numbers	Pages 4 – 5 #4, 6, 14, 20, 22, #30, 34, 40, 46, 52			
1.2 Absolute Value	Pages 9 – 10 #10, 22, 26, 30, 38, #40, 44, 48, 54, 60			
1.3 Solving Equations	Pages 17 – 18 #10, 18, 22, 32, 38, #48, 52, 60, 68, 74			

1.4 Rectangular Coordinates	Pages 26 – 30 #4, 8, 12, 16, 18, #20, 22, 25, 30, 31			

Note: Math Faculty will be available for assistance during summer school (July 15 through August 15, Tuesdays and Wednesdays 8:30 – 10:30 by appointment.

In preparation for our Pre Calculus course, it is essential to have a graphing calculator. It is highly recommended to purchase a Texas Instruments TI-84 Graphing Calculator as this will be the calculator used in class.

CHAPTER 1

Fundamentals

Real numbers, equations, graphs—these topics set the stage for our work in precalculus. How much from previous courses should you remember about solving equations? Section 1.3 provides a review of the fundamentals. The rest of the chapter reviews and begins to extend what you've learned in previous courses about graphs and graphing. For example, we use graphs to visualize trends in

- Spending by the television networks to broadcast the Olympic Games (Exercise 21, page 27)
- Internet usage (Exercise 23, pages 27–28)
- Carbon dioxide levels in the atmosphere (Example 5, page 25)
- U.S. population growth (Exercises 7 and 8 on page 53)

1.1 SETS OF REAL NUMBERS

Natural numbers have been used since time immemorial; fractions were employed by the ancient Egyptians as early as 1700 B.C.; and the Pythagoreans, in ancient Greece, about 400 B.C., discovered numbers, like $\sqrt{2}$, which cannot be fractions. —Stefan Drobot in *Real Numbers* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1964)

What secrets lie hidden in decimals? —Stephan P. Richards in *A Number for Your Thoughts* (New Providence, N.J.: S. P. Richards, 1982)

Here, as in your previous mathematics courses, most of the numbers we deal with are real numbers. These are the numbers used in everyday life, in the sciences, in industry, and in business. Perhaps the simplest way to define a real number is this: A **real number** is any number that can be expressed in decimal form. Some examples of real numbers are

$$7\ (= 7.000\ldots)$$
$$\sqrt{2}\ (= 1.4142\ldots)$$
$$-2/3\ (= -0.\overline{6})$$

(Recall that the bar above the 6 in the decimal $-0.\overline{6}$ indicates that the 6 repeats indefinitely.)

Certain sets of real numbers are referred to often enough to be given special names. These are summarized in the box that follows.

As you've seen in previous courses, the real numbers can be represented as points on a *number line*, as shown in Figure 1. As indicated in Figure 1, the point associated with the number zero is referred to as the **origin.**

The fundamental fact here is that there is a **one-to-one correspondence** between the set of real numbers and the set of points on the line. This means that each real number is identified with exactly one point on the line; conversely, with each point on the line we identify exactly one real number. The real number associated with a given point is called the **coordinate** of the point. As a practical matter, we're

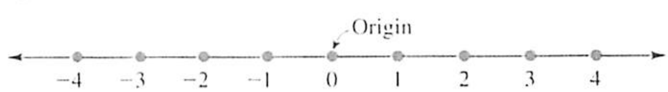


Figure 1

PROPERTY SUMMARY Sets of Real Numbers

Name	Definition and Comments	Examples
Natural numbers	These are the ordinary counting numbers: 1, 2, 3, and so on.	1, 4, 29, 1066
Integers	These are the natural numbers along with their negatives and zero.	−26, 0, 1, 1776
Rational numbers	As the name suggests, these are the real numbers that are *ratios* of two integers (with nonzero denominators). It can be proved that a real number is rational if and only if its decimal expansion *terminates* (e.g., 3.15) or *repeats* (e.g., $2.\overline{43}$).	$4 (= \frac{4}{1})$, $-\frac{2}{3}$, $1.7 (= \frac{17}{10})$, $4.\overline{3}$, $4.1\overline{73}$
Irrational numbers	These are the real numbers that are not rational. Section A.3 of the Appendix contains a proof of the fact that the number $\sqrt{2}$ is irrational. The proof that π is irrational is more difficult. The first person to prove that π is irrational was the Swiss mathematician J. H. Lambert (1728–1777).	$\sqrt{2}$, $3 + \sqrt{2}$, $3\sqrt{2}$, π, $4 + \pi$, 4π

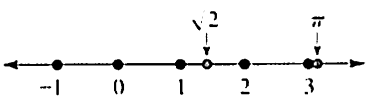


Figure 2

usually more interested in relative locations than precise locations on a number line. For instance, since π is approximately 3.1, we show π slightly to the right of 3 in Figure 2. Similarly, since $\sqrt{2}$ is approximately 1.4, we show $\sqrt{2}$ slightly less than halfway from 1 to 2 in Figure 2.

It is often convenient to use number lines that show reference points other than the integers used in Figure 2. For instance, Figure 3(a) displays a number line with reference points that are multiples of π. In this case it is the integers that we then locate approximately. For example, in Figure 3(b) we show the approximate location of the number 1 on such a line.

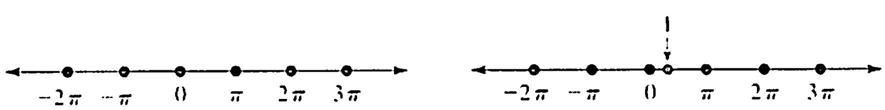
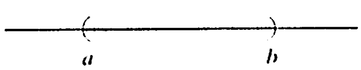


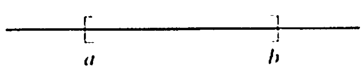
Figure 3

Two of the most basic relations for real numbers are **less than** and **greater than,** symbolized by < and >, respectively. For ease of reference, we review these and two related symbols in the box on page 3.

In general, relationships involving real numbers and any of the four symbols <, ≤, >, and ≥ are called **inequalities.** One of the simplest uses of inequalities is in defining certain sets of real numbers called *intervals.* Roughly speaking, any uninterrupted portion of the number line is referred to as an **interval.** In the definitions that follow, you'll see notations such as $a < x < b$. This means that *both* of the inequalities $a < x$ and $x < b$ hold; in other words, the number x is between a and b.



(a) The open interval (a, b) contains all real numbers from a to b, excluding a and b.



(b) The closed interval $[a, b]$ contains all real numbers from a to b, including a and b.

Figure 4

DEFINITION Open Intervals and Closed Intervals

The **open interval** (a, b) consists of all real numbers x such that $a < x < b$. See Figure 4(a).

The **closed interval** $[a, b]$ consists of all real numbers x such that $a \le x \le b$. See Figure 4(b).

PROPERTY SUMMARY Notation for Less Than and Greater Than

Notation	Definition	Examples
$a < b$	a is less than b. On a number line, oriented as in Figure 1, 2, or 3, the point a lies to the left of b.	$2 < 3$; $-4 < 1$
$a \le b$	a is less than or equal to b.	$2 \le 3$; $3 \le 3$
$b > a$	b is greater than a. On a number line oriented as in Figure 1, 2, or 3, the point b lies to the right of a. ($b > a$ is equivalent to $a < b$.)	$3 > 2$; $0 > -1$
$b \ge a$	b is greater than or equal to a.	$3 \ge 2$; $3 \ge 3$

Note that the brackets in Figure 4(b) are used to indicate that the numbers a and b are included in the interval $[a, b]$, whereas the parentheses in Figure 4(a) indicate that a and b are excluded from the interval (a, b). At times you'll see notation such as $[a, b)$. This stands for the set of all real numbers x such that $a \le x < b$. Similarly, $(a, b]$ denotes the set of all numbers x such that $a < x \le b$.

EXAMPLE 1 Understanding interval notation

Show each interval on a number line, and specify inequalities describing the numbers x in each interval.

$$[-1, 2] \quad (-1, 2) \quad (-1, 2] \quad [-1, 2)$$

SOLUTION
See Figure 5.

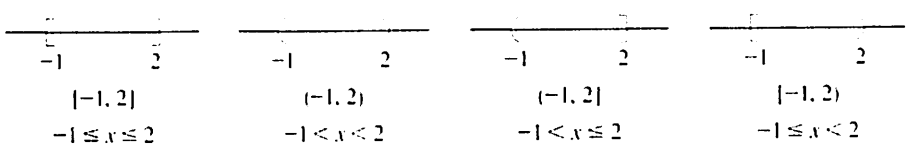


Figure 5

In addition to the four types of intervals shown in Figure 5, we can also consider **unbounded intervals.** These are intervals that extend indefinitely in one direction or the other, as shown, for example, in Figure 6. We also have a convenient notation for unbounded intervals; for example, we indicate the unbounded interval in Figure 6 with the notation $(2, \infty)$.

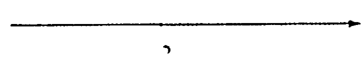
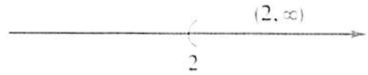
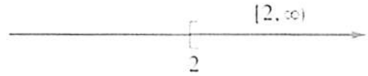
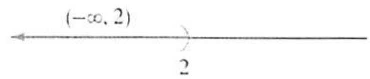
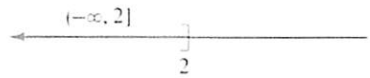
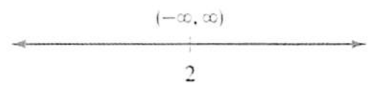


Figure 6

COMMENT AND CAUTION The symbol ∞ is read *infinity*. It is not a real number, and its use in the context $(2, \infty)$ is only to indicate that the interval has no right-hand boundary. In the box that follows we define the five types of unbounded intervals. Note that the last interval, $(-\infty, \infty)$, is actually the entire real-number line.

PROPERTY SUMMARY Unbounded Intervals

For a real number a the notations for unbounded intervals are:

Notation	Defining Inequality	Example
(a, ∞)	$x > a$	
$[a, \infty)$	$x \geq a$	
$(-\infty, a)$	$x < a$	
$(-\infty, a]$	$x \leq a$	
$(-\infty, \infty)$		

EXAMPLE 2 Understanding notation for unbounded intervals

Indicate each set of real numbers on a number line:
(a) $(-\infty, 4]$; (b) $(-3, \infty)$.

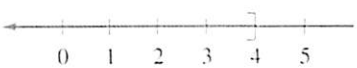


Figure 7

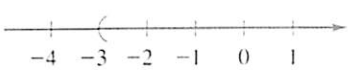


Figure 8

SOLUTION

(a) The interval $(-\infty, 4]$ consists of all real numbers that are less than or equal to 4. See Figure 7.

(b) The interval $(-3, \infty)$ consists of all real numbers that are greater than -3. See Figure 8.

We conclude this section by mentioning that our treatment of the real-number system has been rather informal, and we have not derived any of the rules of arithmetic and algebra using the most basic properties of the real numbers. However, we do list those basic properties and derive some of their consequences in Section A.2 of the Appendix.

EXERCISE SET 1.1

A

In Exercises 1–10, determine whether the number is a natural number, an integer, a rational number, or an irrational number. (Some numbers fit in more than one category.) The following facts will be helpful in some cases: Any number of the form $\sqrt{n}$, where n is a natural number that is not a perfect square, is irrational. Also, the sum, difference, product, and quotient of an irrational number and a nonzero rational are all irrational. (For example, the following four numbers are irrational: $\sqrt{6}$, $\sqrt{10} - 2$, $3\sqrt{15}$, and $-5\sqrt{3}/2$.)

1. (a) -203 (b) $203/2$

2. (a) $27/4$ (b) $\sqrt{27/4}$

3. (a) 10^6 (b) $10^6/10^7$

4. (a) 8.7 (b) $8.\bar{7}$

5. (a) 8.74 (b) $8.\overline{74}$

6. (a) $\sqrt{99}$ (b) $\sqrt{99} + 1$

7. $3\sqrt{101} + 1$
8. $(3 - \sqrt{2}) + (3 + \sqrt{2})$
9. $(\sqrt{5} + 1)/4$
10. $(0.1234)/(0.5677)$

In each of Exercises 11–20, draw a number line similar to the one shown in Figure 1. Then indicate the approximate location of the given number. Where necessary, make use of the approximations $\sqrt{2} \approx 1.4$ and $\sqrt{3} \approx 1.7$. (The symbol $\approx$ means is approximately equal to.*)*

11. $11/4$
12. $-7/8$
13. $1 + \sqrt{2}$
14. $1 - \sqrt{2}$
15. $\sqrt{2} - 1$
16. $-\sqrt{2} - 1$
17. $\sqrt{2} + \sqrt{3}$
18. $\sqrt{2} - \sqrt{3}$
19. $(1 + \sqrt{2})/2$
20. $(2\sqrt{3} + 1)/2$

In Exercises 21–30, draw a number line similar to the one shown in Figure 3(a). Then indicate the approximate location of the given number.

21. $\pi/2$
22. $3\pi/2$
23. $\pi/6$
24. $7\pi/4$
25. -1
26. 3
27. $\pi/3$
28. $3/2$
29. $2\pi + 1$
30. $2\pi - 1$

In Exercises 31–40, say whether the statement is TRUE *or* FALSE. *(In Exercises 37–40, do not use a calculator or table; use instead the approximations $\sqrt{2} \approx 1.4$ and $\pi \approx 3.1$.)*

31. $-5 < -50$
32. $0 < -1$
33. $-2 \leq -2$
34. $\sqrt{7} - 2 \geq 0$
35. $\frac{13}{14} > \frac{15}{16}$
36. $0.\bar{7} > 0.7$
37. $2\pi < 6$
38. $2 \leq (\pi + 1)/2$
39. $2\sqrt{2} \geq 2$
40. $\pi^2 < 12$

In Exercises 41–54, express each interval using inequality notation and show the given interval on a number line.

41. $(2, 5)$
42. $(-2, 2)$
43. $[1, 4]$
44. $[-\frac{3}{2}, \frac{1}{2}]$
45. $[0, 3)$
46. $(-4, 0]$
47. $(-3, \infty)$
48. $(\sqrt{2}, \infty)$
49. $[-1, \infty)$
50. $[0, \infty)$
51. $(-\infty, 1)$
52. $(-\infty, -2)$
53. $(-\infty, \pi]$
54. $(-\infty, \infty)$

B

55. The value of the irrational number π, correct to ten decimal places (without rounding off), is 3.1415926535. By using a calculator, determine to how many decimal places each of the following quantities agrees with π.
   - **(a)** $(4/3)^4$: This is the value used for π in the Rhind papyrus, an ancient Babylonian text written about 1650 B.C.
   - **(b)** $22/7$: Archimedes (287–212 B.C.) showed that $223/71 < \pi < 22/7$. The use of the approximation $22/7$ for π was introduced to the Western world through the writings of Boethius (ca. 480–520), a Roman philosopher, mathematician, and statesman. Among all fractions with numerators and denominators less than 100, the fraction $22/7$ is the best approximation to π.
   - **(c)** $355/113$: This approximation of π was obtained in fifth-century China by Zu Chong-Zhi (430–501) and his son. According to David Wells in *The Penguin Dictionary of Curious and Interesting Numbers* (Harmondsworth, Middlesex, England: Viking Penguin, Ltd., 1986), "This is the best approximation of any fraction below 103993/33102."
   - **(d)** $\frac{63}{25}\left(\frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}}\right)$: This approximation for π was obtained by the Indian mathematician Scrinivasa Ramanujan (1887–1920).

Remark: A simple approximation that agrees with π through the first 14 decimal places is $\frac{355}{113}\left(1 - \frac{0.0003}{3533}\right)$. This approximation was also discovered by Ramanujan. For a fascinating account of the history of π, see the book by Petr Beckmann, *A History of π*, 16th ed. (New York: Barnes & Noble Books, 1989), and for a more modern look at π, see Richard Preston's article, "The Mountains of Pi," in *The New Yorker* (March 2, 1992, pp. 36–67).

In Exercises 56–58, give an example of irrational numbers a and b such that the indicated expression is **(a)** *rational and* **(b)** *irrational.*

56. $a + b$
57. ab
58. a/b

59. **(a)** Give an example in which the result of raising a rational number to a rational power is an irrational number.
   (b) Give an example in which the result of raising an irrational number to a rational power is a rational number.
60. Can an irrational number raised to an irrational power yield an answer that is rational? This problem shows that the answer is "yes." (However, if you study the following solution very carefully, you'll see that even though we've answered the question in the affirmative, we've not pinpointed the specific case in which an irrational number raised to an irrational power is rational.)
   (a) Let $A = (\sqrt{2})^{\sqrt{2}}$. Now, either A is rational or A is irrational. If A is rational, we are done. Why?
   (b) If A is irrational, we are done. Why?
   Hint: Consider $A^{\sqrt{2}}$.

Remark: For more about this problem and related questions, see the article "Irrational Numbers," by J. P. Jones and S. Toporowski in *American Mathematical Monthly*, vol. 80 (1973), pp. 423–424.

1.2 ABSOLUTE VALUE

There has been a real need in analysis for a convenient symbolism for "absolute value" . . . and the two vertical bars introduced in 1841 by Weierstrass, as in $|z|$, have met with wide adoption; . . .—Florian Cajori in *A History of Mathematical Notations*, vol. 1 (La Salle, Ill.: The Open Court Publishing Co., 1928)

As an aid in measuring distances on the number line, we review the concept of *absolute value*. We begin with a definition of absolute value that is geometric in nature. Then, after you have developed some familiarity with the concept, we explain a more algebraic approach that is often useful in analytical work.

DEFINITION **Absolute Value (geometric version)**

The **absolute value** of a real number x, denoted by $|x|$, is the distance from x to the origin.

For instance, because the numbers 5 and -5 are both five units from the origin, we have $|5| = 5$ and $|-5| = 5$. Here are three more examples:

$$|17| = 17 \qquad |-2/3| = 2/3 \qquad |0| = 0$$

EXAMPLE 1 **Evaluating expressions containing absolute values**

Evaluate each expression:
(a) $5 - |6 - 7|$; **(b)** $||-2| - |-3||$.

SOLUTION

(a) $5 - |6 - 7| = 5 - |-1| = 5 - 1 = 4$

(b) $||-2| - |-3|| = |2 - 3| = |-1| = 1$

As we said at the beginning of this section, there is an equivalent, more algebraic way to define absolute value. According to this equivalent definition, the value of $|x|$ is x itself when $x \geq 0$, and the value of $|x|$ is $-x$ when $x < 0$. We can write this symbolically as follows:

DEFINITION **Absolute Value (algebraic version)**

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

EXAMPLE
$|-7| = -(-7) = 7$

By looking at examples with specific numbers, you should be able to convince yourself that both definitions yield the same result. We use the algebraic definition of absolute value in Examples 2 and 3.

EXAMPLE 2 **Rewriting expressions to eliminate absolute value**

Rewrite each expression in a form that does not contain absolute values:
(a) $|\pi - 4| + 1$; **(b)** $|x - 5|$, given that $x \geq 5$; **(c)** $|t - 5|$, given that $t < 5$.

SOLUTION

(a) The quantity $\pi - 4$ is negative (since $\pi \approx 3.14$), and therefore its absolute value is equal to $-(\pi - 4)$. In view of this, we have

$$|\pi - 4| + 1 = -(\pi - 4) + 1 = -\pi + 5$$

(b) Since $x \geq 5$, the quantity $x - 5$ is nonnegative, and therefore its absolute value is equal to $x - 5$ itself. Thus we have

$$|x - 5| = x - 5 \qquad \text{when } x \geq 5$$

(c) Since $t < 5$, the quantity $t - 5$ is negative. Therefore its absolute value is equal to $-(t - 5)$, which in turn is equal to $5 - t$. In view of this, we have

$$|t - 5| = 5 - t \qquad \text{when } t < 5$$

EXAMPLE 3 Simplifying an expression containing absolute values

Simplify the expression $|x - 1| + |x - 2|$, given that x is in the open interval $(1, 2)$.

SOLUTION

Since x is greater than 1, the quantity $x - 1$ is positive, and consequently,

$$|x - 1| = x - 1$$

On the other hand, we are also given that x is less than 2. Therefore the quantity $x - 2$ is negative, and we have

$$|x - 2| = -(x - 2) = -x + 2$$

Putting things together now, we can write

$$\begin{aligned} |x - 1| + |x - 2| &= (x - 1) + (-x + 2) \\ &= -1 + 2 = 1 \end{aligned}$$

In the box that follows, we list several basic properties of the absolute value. Each of these properties can be derived from the definitions. (With the exception of the *triangle inequality*, we shall omit the derivations. For a proof of the triangle inequality, see Exercise 67.)

PROPERTY SUMMARY Properties of Absolute Value

1. For all real numbers x, we have
   (a) $|x| \geq 0$;
   (b) $x \leq |x|$ and $-x \leq |x|$.
   (c) $|x|^2 = x^2$.
2. For all real numbers a and b, we have
   (a) $|ab| = |a||b|$ and $|a/b| = |a|/|b|$ $(b \neq 0)$;
   (b) $|a + b| \leq |a| + |b|$ (the triangle inequality).

EXAMPLE 4 Rewriting an expression to eliminate absolute value

Write the expression $|-2 - x^2|$ in an equivalent form that does not contain absolute values.

SOLUTION

Note that x^2 is nonnegative for any real number x, so $2 + x^2$ is positive. Then $-2 - x^2 = -(2 + x^2)$ is negative. Thus

$$\begin{aligned} |-2 - x^2| &= -(-2 - x^2) && \text{using the algebraic definition of absolute value} \\ &= 2 + x^2 \end{aligned}$$

Alternatively,

$$\begin{aligned} |-2 - x^2| &= |-1(2 + x^2)| \\ &= |-1||2 + x^2| && \text{using Property 2(a)} \\ &= 2 + x^2 \end{aligned}$$

Distance = $|5 - 7| = |7 - 5| = 2$

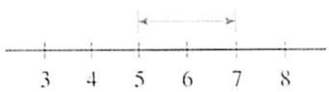


Figure 1

If we think of the real numbers as points on a number line, the distance between two numbers a and b is given by the absolute value of their difference. For instance, as indicated in Figure 1, the distance between 5 and 7, namely, 2, is given by either $|5 - 7|$ or $|7 - 5|$. For reference, we summarize this simple but important fact as follows.

PROPERTY SUMMARY Distance on a Number Line

For real numbers a and b, the **distance** between a and b is $|a - b| = |b - a|$.

EXAMPLE 5 Using absolute value to rewrite statements regarding distance

Rewrite each of the following statements using absolute value notation:

(a) The distance between 12 and -5 is 17.
(b) The distance between x and 2 is 4.
(c) The distance between x and 2 is less than 4.
(d) The number t is more than five units from the origin.

SOLUTION

(a) $|12 - (-5)| = 17$ or $|-5 - 12| = 17$
(b) $|x - 2| = 4$ or $|2 - x| = 4$
(c) $|x - 2| < 4$ or $|2 - x| < 4$
(d) $|t| > 5$

EXAMPLE 6 Displaying intervals defined by absolute value inequalities

In each case, the set of real numbers satisfying the given inequality is one or more intervals on the number line. Show the interval(s) on a number line.

(a) $|x| < 2$ **(b)** $|x| > 2$ **(c)** $|x - 3| < 1$ **(d)** $|x - 3| \geq 1$

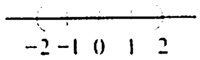


Figure 2
$|x| < 2$

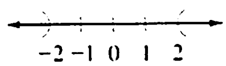


Figure 3
$|x| > 2$

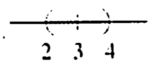


Figure 4
$|x - 3| < 1$

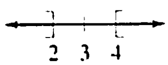


Figure 5
$|x - 3| \geq 1$

SOLUTION

(a) The given inequality tells us that the distance from x to the origin is less than two units. So, as indicated in Figure 2, the number x must lie in the open interval $(-2, 2)$.

(b) The condition $|x| > 2$ means that x is more than two units from the origin. Thus, as indicated in Figure 3, the number x lies either to the right of 2 or to the left of -2.

(c) The given inequality tells us that x must be less than one unit away from 3 on the number line. Looking one unit to either side of 3, then, we see that x must lie between 2 and 4 and x cannot equal 2 or 4. See Figure 4.

(d) The inequality $|x - 3| \geq 1$ says that x is at least one unit away from 3 on the number line. This means that either $x \geq 4$ or $x \leq 2$, as shown in Figure 5. [Here's an alternative way of thinking about this: The numbers satisfying the given inequality are precisely those numbers that do *not* satisfy the inequality in part (c). So for part (d), we need to shade that portion of the number line that was not shaded in part (c).]

EXERCISE SET 1.2

A

In Exercises 1–16, evaluate each expression.

1. $|3|$
2. $3 + |-3|$
3. $|-6|$
4. $-6 - |-6|$
5. $|-1 + 3|$
6. $|-6 + 3|$
7. $|-\frac{4}{5}| - \frac{4}{5}$
8. $|\frac{4}{5}| - \frac{4}{5}$
9. $|-6 + 2| - |4|$
10. $|-3 - 4| - |-4|$
11. $|-8| + |-9|$
12. $|-8| - |-9|$
13. $\left|\frac{27 - 5}{5 - 27}\right|$
14. $\frac{|27 - 5|}{|5 - 27|}$
15. $|7(-8)| - |7| \cdot |-8|$
16. $|(-7)^2| + |-7|^2 - (-|-3|)^3$

In Exercises 17–24, evaluate each expression, given that $a = -2$, $b = 3$, and $c = -4$.

17. $|a - b|^2$
18. $a^2 - |bc|$
19. $|c| - |b| - |a|$
20. $|b + c| - |b| - |c|$
21. $|a + b|^2 - |b + c|^2$
22. $\frac{|a| + |b| + |c|}{|a + b + c|}$
23. $\frac{a + b + |a - b|}{2}$
24. $\frac{a + b - |a - b|}{2}$

In Exercises 25–38, rewrite each expression without using absolute value notation.

25. $|\sqrt{2} - 1| - 1$
26. $|1 - \sqrt{2}| + 1$
27. $|x - 3|$ given that $x \geq 3$
28. $|x - 3|$ given that $x < 3$
29. $|t^2 + 1|$
30. $|x^4 + 1|$
31. $|-\sqrt{3} - 4|$
32. $|-\sqrt{3} - \sqrt{5}|$
33. $|x - 3| + |x - 4|$ given that $x < 3$
34. $|x - 3| + |x - 4|$ given that $x > 4$
35. $|x - 3| + |x - 4|$ given that $3 < x < 4$
36. $|x - 3| + |x - 4|$ given that $x = 4$
37. $|x + 1| + 4|x + 3|$ given that $-\frac{5}{2} < x < -\frac{3}{2}$
38. $|x + 1| + 4|x + 3|$ given that $x < -3$

In Exercises 39–48, rewrite each statement using absolute value notation, as in Example 5.

39. The distance between x and 1 is 1/2.
40. The distance between x and 1 is less than 1/2.
41. The distance between x and 1 is at least 1/2.
42. The distance between x and 1 exceeds 1/2.
43. The distance between y and -4 is less than 1.
44. The distance between x^3 and -1 is at most 0.001.
45. The number y is less than three units from the origin.
46. The number y is less than one unit from the number t.
47. The distance between x^2 and a^2 is less than M.
48. The sum of the distances of a and b from the origin is greater than or equal to the distance of $a + b$ from the origin.

In Exercises 49–60, the set of real numbers satisfying the given inequality is one or more intervals on the number line. Show the interval(s) on a number line.

49. $|x| < 4$
50. $|x| < 2$
51. $|x| > 1$
52. $|x| > 0$
53. $|x - 5| < 3$
54. $|x - 4| < 4$
55. $|x - 3| \leq 4$
56. $|x - 1| \leq \frac{1}{2}$
57. $|x + \frac{1}{3}| < \frac{3}{2}$
58. $|x + \frac{5}{2}| > 1$
59. $|x - 5| \geq 2$
60. $|x + 5| \geq 2$

B

61. In parts (a) and (b), sketch the interval or intervals corresponding to the given inequality:
(a) $|x - 2| < 1$;
(b) $0 < |x - 2| < 1$.
(c) In what way do your answers in (a) and (b) differ? (The distinction is important in the study of *limits* in calculus.)

62. Show that for all real numbers a and b, we have

$$|a| - |b| \leq |a - b|$$

Hint: Beginning with the identity $a = (a - b) + b$, take the absolute value of each side and then use the triangle inequality.

63. Show that

$$|a + b + c| \leq |a| + |b| + |c|$$

for all real numbers a, b, and c. *Hint*: The left-hand side can be written $|a + (b + c)|$. Now use the triangle inequality.

64. Explain why there are no real numbers that satisfy the equation $|x^2 + 4x| = -12$.

C

65. (As background for this exercise, you might want to work Exercise 23.) Prove that

$$\max(a, b) = \frac{a + b + |a - b|}{2}$$

Hint: Consider three separate cases: $a = b$; $a > b$; and $b > a$.

66. (As background for this exercise, you might want to work Exercise 24.) Prove that

$$\min(a, b) = \frac{a + b - |a - b|}{2}$$

67. Complete the following steps to prove the triangle inequality.
(a) Let a and b be real numbers. Which property in the summary box on page 7 tells us that $a \leq |a|$ and $b \leq |b|$?
(b) Add the two inequalities in part (a) to obtain $a + b \leq |a| + |b|$.
(c) In a similar fashion, add the two inequalities $-a \leq |a|$ and $-b \leq |b|$ and deduce that $-(a + b) \leq |a| + |b|$.
(d) Why do the results in parts (b) and (c) imply that $|a + b| \leq |a| + |b|$?

1.3 SOLVING EQUATIONS (REVIEW AND PREVIEW)

I learned algebra fortunately by not learning it at school, and knowing that the whole idea was to find out what x was, and it didn't make any difference how you did it. —Physicist Richard Feynman (1918–1988) in Jagdish Mehra's *The Beat of a Different Drum* (New York: Oxford University Press, 1994)

The title of al-Khwarizmi's second and most important book, Hisab al-jabr w'al muqabala [830] *. . . has given us the word* algebra. Al-jabr *means transposing a quantity from one side of an equation to the other, while* muqabala *signifies the simplification of the resulting equation.* —Stuart Hollingdale in *Makers of Mathematics* (Harmondsworth, Middlesex, England: Penguin Books, Ltd., 1989)

"Algebra is a merry science," Uncle Jakob would say. "We go hunting for a little animal whose name we don't know, so we call it x. When we bag our game we pounce on it and give it its right name." —Physicist Albert Einstein (1879–1955)

Consider the familiar expression for the area of a circle of radius r, namely, πr^2. Here π is a constant; its value never changes throughout the discussion. On the other hand, r is a variable; we can substitute any positive number for r to obtain the area of a particular circle. More generally, by a **constant** we mean either a particular number (such as π, or -17, or $\sqrt{2}$) or a letter with a value that remains fixed (although perhaps unspecified) throughout a given discussion. In contrast, a **variable** is a letter for which we can substitute any number selected from a given set of numbers. The given set of numbers is called the **domain** of the variable.

Some expressions will make sense only for certain values of the variable. For instance, $1/(x - 3)$ will be undefined when x is 3 (for then the denominator is zero). So in this case we would agree that the domain of the variable x consists of all real numbers except $x = 3$. Similarly, throughout this chapter we adopt the following convention.

The Domain Convention

The domain of a variable in a given expression is the set of all real-number values of the variable for which the expression is defined.

It's customary to use the letters near the end of the alphabet for variables; letters from the beginning of the alphabet are used for constants. For example, in the expression $ax + b$, the letter x is the variable and a and b are constants.

EXAMPLE 1 Specifying variables, constants, and the domain in an expression

Specify the variable, the constants, and the domain of the variable for each of the following expressions:

(a) $3x + 4$; **(b)** $\dfrac{1}{(t-1)(t+3)}$; **(c)** $ay^2 + by + c$; **(d)** $4x + 3x^{-1}$.

SOLUTION

	VARIABLE	CONSTANTS	DOMAIN
(a) $3x + 4$	x	3, 4	The set of all real numbers.
(b) $\dfrac{1}{(t-1)(t+3)}$	t	$1, -1, 3$	The set of all real numbers except $t = 1$ and $t = -3$.
(c) $ay^2 + by + c$	y	a, b, c	The set of all real numbers.
(d) $4x + 3x^{-1}$	x	4, 3	The set of all real numbers except $x = 0$.

Note: The number 2 that appears in part (c) above is an exponent; y^2 is shorthand notation for the product $y \times y$. Similarly, in part (d) x^{-1} is shorthand notation for $1/x$.

Now let's review the terminology and skills used in solving two basic types of equations: *linear equations* and *quadratic equations*.

DEFINITION Linear Equation in One Variable

A **linear** or **first-degree equation in one variable** is an equation that can be written in the form

$$ax + b = 0 \qquad \text{with } a \text{ and } b \text{ real numbers and } a \neq 0$$

Here are three examples of linear equations in one variable:

$$2x = 10, \qquad 3m + 1 = 2, \qquad \text{and} \qquad \frac{y}{2} = \frac{y}{3} + 1$$

As with any equation involving a variable, each of these equations is neither true nor false *until* we replace the variable with a number. By a **solution** or a **root** of an equation in one variable, we mean a value for the variable that makes the equation a true statement. For example, the value $x = 5$ is a solution of the equation $2x = 10$, since, with $x = 5$, the equation becomes $2(5) = 10$, which is certainly true. We also say in this case that the value $x = 5$ **satisfies** the equation. To check an equation means to verify that the original equation with the solution substituted for the variable is a true statement.

Equations that become true statements for *all* values in the domain of the variable are called **identities.** Two examples of identities are

$$x^2 - 9 = (x - 3)(x + 3) \qquad \text{and} \qquad \frac{4x^2}{x} = 4x$$

The first is true for all real numbers; the second is true for all real numbers except 0. In contrast to this, a **conditional equation** is true only for some (or perhaps none) of the values of the variable. Two examples of conditional equations are $2x = 10$ and $x = x + 1$. The first of these is true only when $x = 5$. The second equation has no solution (because, intuitively at least, no number can be one more than itself).

We say that two equations are **equivalent** when they have exactly the same solutions. In this section, and throughout the text, the basic method for solving an equation in one variable involves writing a sequence of equivalent equations until we finally reach an equivalent equation of the form

$$\text{variable} = \text{a number}$$

which explicitly displays a solution of the original equation. In generating equivalent equations, we rely on the following three principles. (These can be justified by using the properties of real numbers discussed in Appendix A.2.)

Procedures That Yield Equivalent Equations

1. Adding or subtracting the same quantity on both sides of an equation produces an equivalent equation.
2. Multiplying or dividing both sides of an equation by the same nonzero quantity produces an equivalent equation.
3. Simplifying an expression on either side of an equation produces an equivalent equation.

The examples that follow show how these principles are applied in solving various equations. *Note:* Beginning in Example 2, we use some basic factoring techniques from elementary or intermediate algebra. If you find that you need a quick reference for factoring formulas, see the inside back cover of this book. For a detailed review (with many examples and practice exercises) see Appendix B.4.

EXAMPLE 2 Solving equations equivalent to linear equations

(a) Solve: $3[1 - 2(x + 1)] = 2 - x$.
(b) Solve: $ax + b = c$; where a, b, and c are constants, $a \neq 0$.
(c) Solve: $\dfrac{1}{x+5} = \dfrac{2}{x-3} + \dfrac{2x+2}{x^2+2x-15}$.

SOLUTION

(a)

$$\begin{aligned}
3[1-2(x+1)] &= 2-x \\
3[1-2x-2)] &= 2-x && \text{simplifying the left-hand side} \\
3(-1-2x) &= 2-x \\
-3-6x &= 2-x \\
-3-5x &= 2 && \text{adding } x \text{ to both sides} \\
-5x &= 5 && \text{adding 3 to both sides} \\
x &= -1 && \text{dividing both sides by } -5
\end{aligned}$$

CHECK Replacing x with -1 in the original equation yields

$$\begin{aligned}
3[1-2(0)] &\stackrel{?}{=} 2-(-1) \\
3(1) &\stackrel{?}{=} 2+1 \qquad \text{True}
\end{aligned}$$

(b)

$$\begin{aligned}
ax+b &= c \\
ax &= c-b && \text{subtracting } b \text{ from both sides} \\
x &= \frac{c-b}{a} && \text{dividing both sides by } a \text{ (recall that } a \neq 0)
\end{aligned}$$

CHECK Replacing x with $\dfrac{c-b}{a}$ in the original equation yields

$$\begin{aligned}
a\left(\frac{c-b}{a}\right)+b &\stackrel{?}{=} c \\
c-b+b &\stackrel{?}{=} c \\
c &\stackrel{?}{=} c \qquad \text{True}
\end{aligned}$$

(c) A common strategy in solving equations with fractions is to multiply through by the least common denominator. This eliminates the need to work with fractions. By factoring the denominator $x^2+2x-15$, we obtain

$$\frac{1}{x+5} = \frac{2}{x-3} + \frac{2x+2}{(x+5)(x-3)}$$

From this we see that the least common denominator for the three fractions is $(x+5)(x-3)$. Now, multiplying both sides by this least common denominator, we have

$$\frac{(x+5)(x-3)}{x+5} = \frac{2(x+5)(x-3)}{x-3} + \frac{(2x+2)(x+5)(x-3)}{(x+5)(x-3)}$$

$$\begin{aligned}
x-3 &= 2(x+5)+2x+2 \\
x-3 &= 2x+10+2x+2 && \text{simplifying} \\
x-3 &= 4x+12 && \text{simplifying} \\
-3x-3 &= 12 && \text{subtracting } 4x \text{ from both sides} \\
-3x &= 15 && \text{adding 3 to both sides} \\
x &= -5 && \text{dividing both sides by } -3
\end{aligned}$$

CHECK The preceding steps show that *if* the equation has a solution, then the solution is $x = -5$. With $x = -5$, however, the left-hand side of the original equation becomes $1/(-5+5)$, or $1/0$, which is undefined. We conclude therefore that the given equation has no solution.

□

In Example 2(c) the value $x = -5$, which does not check in the original equation, is called an **extraneous root** or **extraneous solution.** How is it that an extraneous solution was generated in Example 2(c)? We multiplied both sides by $(x + 5)(x - 3)$. Since we didn't know at that stage whether the quantity $(x + 5)(x - 3)$ was nonzero, we could not be certain that the resulting equation was actually an equivalent equation. [Indeed, as it turns out with $x = -5$, the quantity $(x + 5)(x - 3)$ *is* equal to zero.] For this reason, it is always necessary to check in the original equation any solutions you obtain as a result of multiplying both sides of an equation by an expression involving the variable. We restate this advice in the box that follows.

PROPERTY SUMMARY Extraneous Solutions

Multiplying both sides of an equation by an expression involving the variable may introduce extraneous solutions that do not check in the original equation. Therefore, it is always necessary to check any candidates for solutions that you obtain in this manner.

EXAMPLE 3 Solving equations where the unknown is the denominator

Solve the given equation for x:

$$y = \frac{ax + b}{cx + d} \qquad \text{where } cx + d \neq 0,\ yc - a \neq 0$$

SOLUTION

Multiplying both sides of the given equation by the nonzero quantity $cx + d$ yields

$$\begin{aligned} y(cx + d) &= ax + b & & \\ ycx + yd &= ax + b & & \text{simplifying} \\ ycx - ax &= b - yd & & \text{gathering terms involving } x \\ x(yc - a) &= b - yd & & \text{factoring} \\ x &= \frac{b - yd}{yc - a} & & \text{dividing both sides by } yc - a \neq 0 \end{aligned}$$

You should check for yourself that the expression for x on the right-hand side of this last equation indeed satisfies the original equation.

In the example just concluded, we used a basic factoring technique from elementary algebra to solve the equations. Factoring is also useful in solving *quadratic equations.*

DEFINITION Quadratic Equation

A **quadratic equation** is an equation in one variable that can be written in the form

$$ax^2 + bx + c = 0 \qquad \text{with } a, b, \text{ and } c \text{ real numbers and } a \neq 0$$

To solve a quadratic equation by factoring, we rely on the following familiar and important property of the real-number system.

PROPERTY SUMMARY
Zero-Product Property of Real Numbers

$$pq = 0 \quad \text{if and only if} \quad p = 0 \text{ or } q = 0 \quad \text{(or both)}$$

EXAMPLE 4 Applying the zero-product property to solve quadratic equations

Solve:
(a) $8x^2 - 3 = 10x$; (b) $4x^2 - 9 = 0$.

SOLUTION

(a) In preparation for using the zero-product property, we first rewrite the equation so the right-hand side is zero. Then we have

$$8x^2 - 10x - 3 = 0$$

$$(2x - 3)(4x + 1) = 0 \qquad \text{Check the factoring.}$$

$$\begin{array}{r|r} 2x - 3 = 0 & 4x + 1 = 0 \\ x = \dfrac{3}{2} & x = -\dfrac{1}{4} \end{array}$$

You can check that the values $x = 3/2$ and $x = -1/4$ both satisfy the given equation.

(b) Using difference-of-squares factoring, we have

$$(2x - 3)(2x + 3) = 0$$

$$\begin{array}{r|r} 2x - 3 = 0 & 2x + 3 = 0 \\ x = \dfrac{3}{2} & x = -\dfrac{3}{2} \end{array}$$

You can check that the values $x = 3/2$ and $x = -3/2$ both satisfy the given equation.

Here's another perspective on Example 4(b). Instead of using factoring to solve the equation $4x^2 - 9 = 0$, we can instead rewrite it as $x^2 = 9/4$. Taking the principal square root of both sides then yields

$$\sqrt{x^2} = \sqrt{9/4}$$

and therefore

$$|x| = \frac{3}{2} \quad \text{(Why?)}$$

By looking at this last equation, we can see that there are two solutions, $x = 3/2$ and $x = -3/2$. (Those are the only two numbers with absolute values of $3/2$.) We abbreviate these two solutions by writing $x = \pm 3/2$. In practice, we usually omit showing the step involving the absolute value. For example, to solve the equation $9x^2 - 2 = 0$, just rewrite it as $x^2 = 2/9$. Then "taking square roots" immediately yields the two solutions $x = \pm\sqrt{2/9} = \pm\sqrt{2}/3$.

Not all quadratic equations can be solved by factoring. Consider, for example, the equation $x^2 - 2x - 4 = 0$. The only three possible factorizations with integer coefficients are

$$(x - 4)(x + 1) \qquad (x + 4)(x - 1) \qquad (x - 2)(x + 2)$$

but none yields the appropriate middle term, $-2x$, when multiplied out. In cases such as these, we can use the *quadratic formula*, given in the box that follows. (In Section 2.1, we'll derive this formula and look at some of its implications. For now, though, the focus is simply on using this formula to calculate solutions.)

The Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 5 Using the quadratic formula to solve a quadratic equation

Use the quadratic formula to solve the equation $2x^2 = 3 - 4x$.

SOLUTION

We first rewrite the given equation as $2x^2 + 4x - 3 = 0$, so that it has the form $ax^2 + bx + c = 0$. By comparing these last two equations, we see that $a = 2$, $b = 4$, and $c = -3$. Therefore

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{-4 \pm \sqrt{40}}{4} = \frac{-4 \pm 2\sqrt{10}}{4} = \frac{-2 \pm \sqrt{10}}{2}$$

Thus, the two solutions are $\frac{-2 + \sqrt{10}}{2}$ and $\frac{-2 - \sqrt{10}}{2}$.

The techniques that we've reviewed in this section for solving linear and quadratic equations will be used throughout this book; you'll see applications in analyzing graphs and functions and in solving many types of applied problems. Linear and quadratic equations both fall under the general heading of *polynomial equations*.

DEFINITION Polynomial Equation

A **polynomial equation** in one variable is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where the subscripted letter a's represent constants and the exponents on the variable are nonnegative integers.

EXAMPLES

(a) $4x^2 - 5x - 1 = 0$

(b) $x^3 - 2x^2 - 3x = 0$

(c) $2x^4 - \frac{4}{3}x^3 + x^2 - 3x + \sqrt{2} = 0$

If a_n is not zero, the **degree** of the polynomial equation is the largest exponent of the variable that appears in the equation. For example, the degrees of equations (a), (b), and (c) in the box above are 2, 3, and 4, respectively.

As we've seen in this section, polynomial equations of degree 1 (linear equations) and polynomial equations of degree 2 (quadratic equations) can be solved by using fairly basic algebra. So too can some higher-degree equations. For instance, we can use factoring and the zero-product property to solve equation (b) in the box above. We have

$$x^3 - 2x^2 - 3x = 0$$

$$x(x^2 - 2x - 3) = 0 \qquad \text{factoring out the common factor } x$$

$$x(x - 3)(x + 1) = 0 \qquad \text{factoring the quadratic}$$

Therefore

$$x = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \qquad \text{using the zero-product property}$$

From these last three equations we conclude that the solutions of the third-degree polynomial equation $x^3 - 2x^2 - 3x = 0$ are $x = 0, 3$, and -1. (You should check for yourself that each of these numbers indeed satisfies the equation.)

Unfortunately, not all polynomial equations are as easy to solve as this last one. Chapter 12 contains a more complete discussion of polynomial equations and an answer to the following question: Is there a general formula, similar to the quadratic formula, for solving *any* polynomial equation?

EXERCISE SET 1.3

A

In Exercises 1–5, determine whether the given value is a solution of the equation.

1. $4x - 5 = -13;\ x = -2$
2. $\frac{1}{x} = \frac{3}{x} - 1;\ x = 2$
3. $\frac{2}{y - 1} - \frac{3}{y} = \frac{7}{y^2 - y};\ y = -3$
4. $(y - 1)(y + 5) = 0;\ y = 5$
5. $m^2 + m - \frac{5}{16} = 0;\ m = \frac{1}{4}$
6. Verify that the numbers $1 + \sqrt{5}$ and $1 - \sqrt{5}$ both satisfy the equation $x^2 - 2x - 4 = 0$.

Solve each equation in Exercises 7–19.

7. $2x - 3 = -5$
8. $2m - 1 + 3m + 5 = 6m - 8$
9. $1 - (2m + 5) = -3m$
10. $(x + 2)(x + 1) = x^2 + 11$
11. $t - \{4 - [t - (4 + t)]\} = 6$
12. $\frac{x}{3} + \frac{2x}{5} = \frac{-11}{5}$
13. $1 - \frac{y}{3} = 6$
14. $\frac{x - 1}{4} + \frac{2x + 3}{-1} = 0$
15. $\frac{1}{x} = \frac{4}{x} - 1$
16. $\frac{1}{y} + 1 = \frac{3}{y} - \frac{1}{2y}$
17. $\frac{1}{x - 3} - \frac{2}{x + 3} = \frac{1}{x^2 - 9}$
18. $\frac{1}{x - 5} + \frac{1}{x + 5} = \frac{2x + 1}{x^2 - 25}$
19. $\frac{4}{x + 2} + \frac{1}{x - 2} = \frac{4}{x^2 - 4}$
20. $\frac{3}{2x + 1} - \frac{4}{x + 1} = \frac{2}{2x^2 + 3x + 1}$
21. $\frac{5}{x - 4} - \frac{3}{2x^2 - 5x - 12} = \frac{1}{2x + 3}$
22. (a) $\frac{2}{3x} = \frac{3}{x}$
    (b) $\frac{2}{3x} = \frac{3}{x + 1}$
    (c) $\frac{2}{3x} = \frac{3}{x} + 1$
23. (a) $\frac{3}{x - 2} = \frac{5}{9x}$
    (b) $\frac{3}{x - 2} = \frac{5}{9x - 2}$
    (c) $\frac{3}{x - 2} = \frac{5}{\frac{5}{3}x - 2}$

In Exercises 24–33, solve each equation by factoring.

24. $x^2 - 5x - 6 = 0$
25. $x^2 - 5x = -6$
26. $10z^2 - 13z - 3 = 0$
27. $3t^2 - t - 4 = 0$
28. $(x + 1)^2 - 4 = 0$
29. $x^2 + 3x - 40 = 0$
30. $x(2x - 13) = -6$
31. $x(3x - 23) = 8$
32. $x(x + 1) = 156$
33. $x^2 + (2\sqrt{5})x + 5 = 0$

In Exercises 34–41, use the quadratic formula to solve each equation. In Exercises 34–39, give two forms for each solution: an expression containing a radical and a calculator approximation rounded off to two decimal places.

34. $2x^2 + 3x - 4 = 0$
35. $4x^2 - 3x - 9 = 0$
36. $x(x + 6) = -2$
37. $x(3x + 8) = -2$

38. $2x^2 - 10 = -\sqrt{2}x$

39. $\sqrt{3}x^2 + \sqrt{3} = 6x$

40. $12x^2 - 25x = -12$

41. $24x^2 + 23x = -5$

In Exercises 42–47, solve the equations using any method you choose.

42. $x^2 = 24$

43. $2y^2 - 50 = 0$

44. $\frac{1}{8} - t^2 = 0$

45. $x^2 - \sqrt{5} = 0$

46. (a) $u(u + 18) = -81$
(b) $u(u + 18) = 81$

47. (a) $x^2 + 156x + 5963 = 0$
(b) $144y^2 - 54y = 13$

48. Solve each of the following equations for x. *Hint:* As in the text, begin by factoring out a common factor.
(a) $x^3 - 13x^2 + 42x = 0$
(b) $x^3 - 6x^2 + x = 0$

For Exercises 49–58, solve each equation for x in terms of the other letters.

49. $3ax - 2b = b + 3$

50. $ax + b = bx - a$

51. $ax + b = bx + a$

52. $\frac{x}{a} + \frac{x}{b} = 1$

53. $\frac{1}{x} = a + b$

54. $\frac{1}{ax} = \frac{1}{bx} - \frac{1}{c}$

55. $\frac{1}{a} - \frac{1}{x} = \frac{1}{x} - \frac{1}{b}$

56. (a) $y = mx + b$, where $m \neq 0$
(b) $y - y_1 = m(x - x_1)$, where $m \neq 0$
(c) $\frac{x}{a} + \frac{y}{b} = 1$
(d) $Ax + By + C = 0$, where $A \neq 0$

57. $(ax + b)^2 - (bx + a)^2 = 0$, where $a \neq \pm b$

58. $(x - p)^2 + (x - q)^2 = p^2 + q^2$

B

In Exercises 59–64, solve each equation for x in terms of the other letters.

59. $a^2(a - x) = b^2(b + x) - 2abx$, where $a \neq b$

60. $\frac{b}{ax - 1} - \frac{a}{bx - 1} = 0$, where $a \neq b$

61. $\frac{a - x}{a - b} - 2 = \frac{c - x}{b - c}$

62. $\frac{x + 2p}{2q - x} + \frac{x - 2p}{2q + x} - \frac{4pq}{4q^2 - x^2} = 0$

63. $\frac{x - a}{x - b} = \frac{b - x}{a - x}$, where $a \neq b$

64. $1 - \frac{a}{b}\left(1 - \frac{a}{x}\right) - \frac{b}{a}\left(1 - \frac{b}{x}\right) = 0$

In Exercises 65–68, solve each equation for the indicated variable.

65. $S = 2\pi r^2 + 2\pi rh$; for h

66. $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$; for y

67. $d = \frac{r}{1 + rt}$; for r

68. $S = \frac{rl - a}{r - 1}$; for r

Solve the equations in Exercises 69–74. (In these exercises, you'll need to multiply both sides of the equations by expressions involving the variable. Remember to check your answers in these cases.)

69. $\frac{3}{x + 5} + \frac{4}{x} = 2$

70. $\frac{5}{x + 2} - \frac{2x - 1}{5} = 0$

71. $1 - x - \frac{2}{6x + 1} = 0$

72. $\frac{x^2 - 3x}{x + 1} = \frac{4}{x + 1}$

73. $\frac{x}{x - 2} + \frac{x}{x + 2} = \frac{8}{x^2 - 4}$

74. $\frac{2x}{x^2 - 1} - \frac{1}{x + 3} = 0$

75. Given the equation $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$:
(a) Solve to show $x = \frac{ab}{a + b}$, provided $a + b \neq 0$.
(b) Check the solution.

1.4 RECTANGULAR COORDINATES. VISUALIZING DATA

The name coordinate does not appear in the work of Descartes. This term is due to Leibniz and so are abscissa *and* ordinate *(1692).*—David M. Burton in *The History of Mathematics: An Introduction*, 2nd ed. (Dubuque, Iowa: Wm. C. Brown Publishers, 1991)

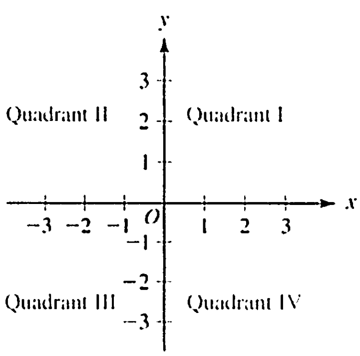
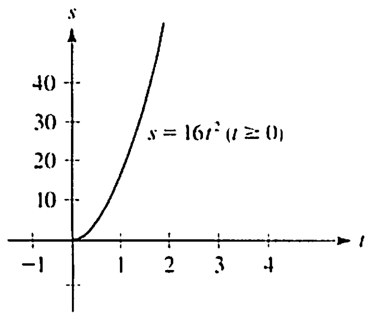


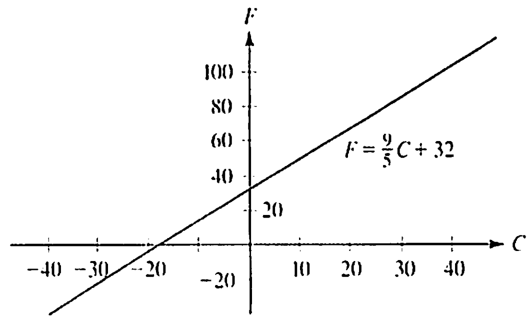
Figure 1

In previous courses you learned to work with a rectangular coordinate system such as that shown in Figure 1. In this section we review some of the most basic formulas and techniques that are useful here.

The point of intersection of the two perpendicular number lines, or **axes**, is called the **origin** and is denoted by the letter O. The horizontal and vertical axes are often labeled the **x-axis** and the **y-axis**, respectively, but any other variables will



(a) A graph of the formula $s = 16t^2$ in a t-s coordinate system. [The formula relates the distance s (in feet) and the time t (in seconds) for an object falling in a vacuum.]



(b) A graph of the equation $F = \frac{9}{5}C + 32$ in a C-F coordinate system. (The equation gives the relationship between the temperature C on the Celsius scale and F on the Fahrenheit scale.)

Figure 2

do just as well for labeling the axes. See Figure 2 for examples of this. (We'll discuss curves or graphs like the ones in Figure 2 in later sections.)

Note that in Figures 1 and 2 the axes divide the plane into four regions, or **quadrants,** labeled I through IV, as shown in Figure 1. Unless indicated otherwise, we assume that the same unit of length is used on both axes. In Figure 1, the same scales are used on both axes; not so in Figure 2.

Now look at the point P in Figure 3(a). Starting from the origin O, one way to reach P is to move three units in the positive x-direction and then two units in the positive y-direction. That is, the location of P relative to the origin and the axes is "right 3, up 2." We say that the **coordinates** of P are (3, 2). The first number within the parentheses conveys the information "right 3," and the second number conveys the information "up 2." We say that the **x-coordinate** of P is 3 and the **y-coordinate** of P is 2. Likewise, the coordinates of point Q in Figure 3(a) are (−2, 4). With this coordinate notation in mind, observe in Figure 3(b) that (3, 2) and (2, 3) represent different points; that is, the order in which the two numbers appear within the parentheses affects the location of the point. Figure 3(c) displays various points with given coordinates; you should check for yourself that the coordinates correspond correctly to the location of each point.

Some terminology and notation: The x-y coordinate system that we have described is often called a **Cartesian coordinate system.** The term *Cartesian* is used in honor of René Descartes, the seventeenth-century French philosopher and mathematician. The coordinates (x, y) of a point P are referred to as an **ordered pair.**

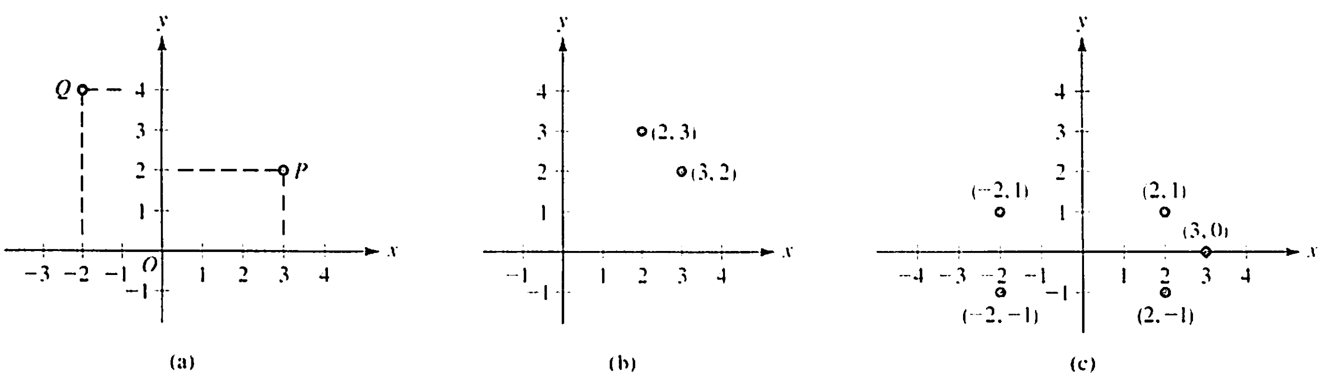


Figure 3

Recall, for example, that (3, 2) and (2, 3) represent different points; that is, the order of the numbers matters. The x-coordinate of a point is sometimes referred to as the **abscissa** of the point; the y-coordinate is the **ordinate.** The notation $P(x, y)$ means that P is a point that has coordinates (x, y). At times, we abbreviate the phrase *the point whose coordinates are* (x, y) to simply *the point* (x, y).

The next part of our work in this section depends on a key result from elementary geometry, the Pythagorean theorem. For reference, we state this theorem and its converse in the box that follows. (For proofs of the Pythagorean theorem, see Exercises 32 and 33 at the end of this section or Exercise 100 in the Chapter Review Exercises.)

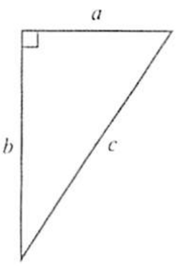


Figure 4

The Pythagorean Theorem and Its Converse

1. Pythagorean Theorem
(See Figure 4.) In a right triangle the lengths of the sides are related by the equation
$$a^2 + b^2 = c^2$$
where a and b are the lengths of the sides forming the right angle and c is the length of the hypotenuse (the side opposite the right angle).
2. Converse
If the lengths a, b, and c of the sides of a triangle are related by an equation of the form $a^2 + b^2 = c^2$, then the triangle is a right triangle, and c is the length of the hypotenuse.

EXAMPLE 1 Using the Pythagorean theorem to find a distance

Use the Pythagorean theorem to calculate the distance d between the points (2, 1) and (6, 3).

SOLUTION

Figure 5

We plot the two given points and draw a line connecting them, as shown in Figure 5. Then we draw the broken lines as shown, parallel to the axes, and apply the Pythagorean theorem to the right triangle that is formed. The base of the triangle is four units long. You can see this by simply counting spaces or by using absolute value, as discussed in Section 1.2: $|6 - 2| = 4$. The height of the triangle is found to be two units, either by counting spaces or by computing the absolute value: $|3 - 1| = 2$. Thus we have

$$d^2 = 4^2 + 2^2 = 20$$
$$d = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$$

Note: Since d is a distance, we disregard the solution $-\sqrt{20}$ of the equation $d^2 = 20$.

The method we used in Example 1 can be applied to derive a general formula for the distance d between any two points (x_1, y_1) and (x_2, y_2) (see Figure 6).

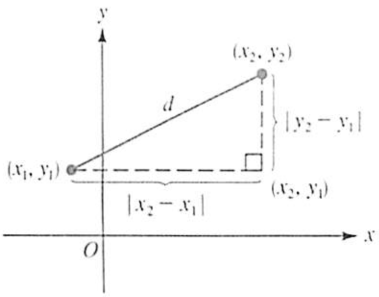


Figure 6

Just as before, we draw in the right triangle and apply the Pythagorean theorem. We have

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \text{(Why?)}$$

and therefore

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This last equation is referred to as the **distance formula.** For reference, we restate it in the box that follows.

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Examples 2–4 demonstrate some simple calculations involving the distance formula.

NOTE In computing the distance between two given points, it does not matter which one you treat as (x_1, y_1) and which as (x_2, y_2), because quantities such as $x_2 - x_1$ and $x_1 - x_2$ are negatives of each other and so have equal squares.

EXAMPLE 2 Using the distance formula

Calculate the distance between the points $(2, -6)$ and $(5, 3)$.

SOLUTION
Substituting $(2, -6)$ for (x_1, y_1) and $(5, 3)$ for (x_2, y_2) in the distance formula, we have

$$d = \sqrt{(5-2)^2 + [3 - (-6)]^2}$$
$$= \sqrt{3^2 + 9^2} = \sqrt{90}$$
$$= \sqrt{9}\sqrt{10} = 3\sqrt{10}$$

You should check for yourself that the same answer is obtained using $(2, -6)$ as (x_2, y_2) and $(5, 3)$ as (x_1, y_1).

EXAMPLE 3 Using the distance formula and the converse of the Pythagorean theorem

Is the triangle with vertices $D(-2, -1)$, $E(4, 1)$, and $F(3, 4)$ a right triangle?

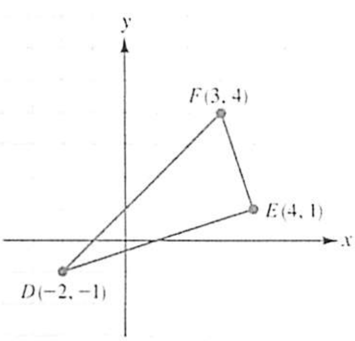


Figure 7

SOLUTION
First we sketch the triangle in question (see Figure 7). From the sketch it appears that angle E could be a right angle, but certainly this is not a proof. We

need to use the distance formula to calculate the lengths of the three sides and then check whether any relation of the form $a^2 + b^2 = c^2$ holds. The calculations are as follows:

$$DE = \sqrt{[4 - (-2)]^2 + [1 - (-1)]^2} = \sqrt{36 + 4} = \sqrt{40}$$
$$EF = \sqrt{(4 - 3)^2 + (1 - 4)^2} = \sqrt{1 + 9} = \sqrt{10}$$
$$DF = \sqrt{[3 - (-2)]^2 + [4 - (-1)]^2} = \sqrt{25 + 25} = \sqrt{50}$$

Because $(\sqrt{40})^2 + (\sqrt{10})^2 = (\sqrt{50})^2$, D, E, and F are vertices of a right triangle with hypotenuse DF and right angle at vertex E, and so $\triangle DEF$ is a right triangle. (In Section 1.6 you'll see that this result can be obtained more easily by using the concept of slope.)

EXAMPLE 4 Using the distance formula to find a radius

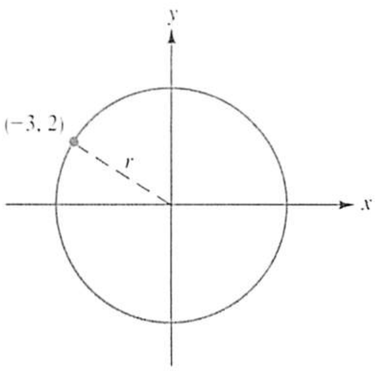


Figure 8

(a) Find the radius r of the circle in Figure 8. (Assume that the center of the circle is located at the origin.)

(b) Compute the area and the circumference of the circle. For each answer, give exact expressions and also calculator approximations rounded to one decimal place.

SOLUTION

(a) The radius r is the distance from center $(0, 0)$ to the given point $(-3, 2)$ on the circle. Using the distance formula, we have

$$\begin{aligned} r &= \sqrt{(-3 - 0)^2 + (2 - 0)^2} \\ &= \sqrt{9 + 4} = \sqrt{13} \text{ units} \end{aligned}$$

(b) Recall the formulas for the area A and the circumference C of a circle of radius r: $A = \pi r^2$ and $C = 2\pi r$. Using the value for r from part (a), we have

$$\begin{aligned} A = \pi r^2 &= \pi(\sqrt{13})^2 \\ &= 13\pi \text{ square units} \\ &\approx 40.8 \text{ square units} \end{aligned} \qquad \begin{aligned} C = 2\pi r &= 2\pi\sqrt{13} \text{ units} \\ &\approx 22.7 \text{ units} \end{aligned}$$

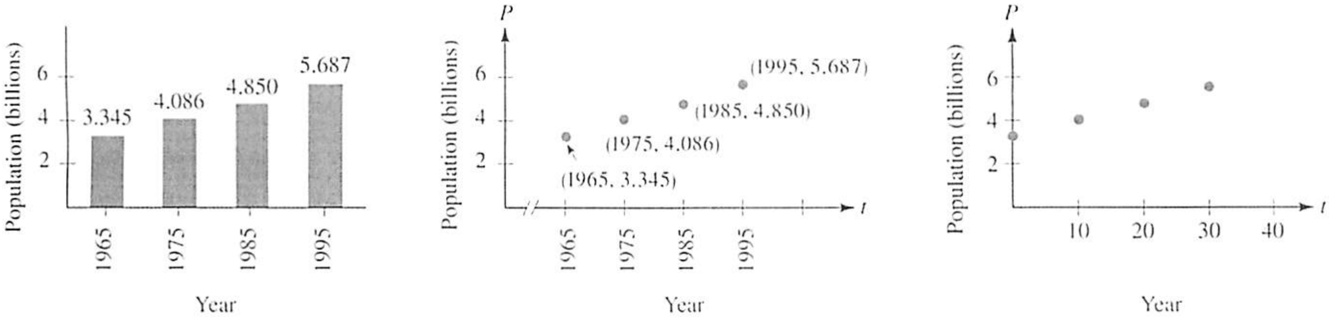
One of the important applications of rectangular coordinates is in displaying quantitative data. You see instances of this every day in newspapers, in magazines, and in textbooks as diverse as astronomy to zoology. We show some examples in the figures and discussion that follow.

Table 1 provides world population data for the period 1965–1995. In Figure 9(a) the familiar *bar graph* (or *column chart*) format is used to display the data from the table. In Figure 9(b) we've plotted the data in a rectangular coordinate system. On the horizontal axis the variable t represents years; the variable P represents population in units of one billion. The data in the first row of the table (which state that the population in 1965 was 3.345 billion) are plotted in Figure 9(b) as the point (1965, 3.345). Likewise, the second row of data in the table gives us the point (1975, 4.086), and so on. Sometimes it is more convenient to work with smaller numbers on the horizontal axis than those used in Figure 9(b). One very common way to do this is indicated in Figure 9(c), where we are now letting the variable t

TABLE 1 World Population 1965–1995

Year	Population (billions)
1965	3.345
1975	4.086
1985	4.850
1995	5.687

Source: U.S. Census Bureau (International Data Base)



(a) Bar graph or column chart

(b) Rectangular coordinates with the variable t representing the year

(c) Rectangular coordinates with the variable t representing the number of years since 1965

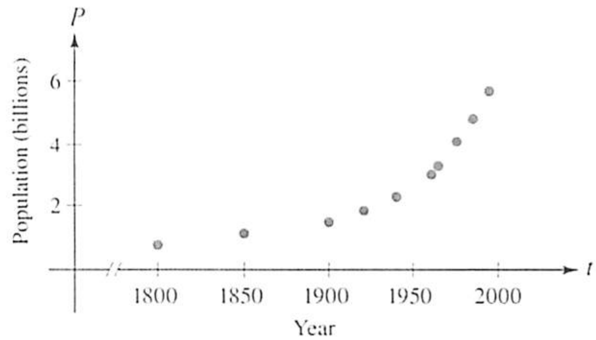
Figure 9
World population 1965–1995

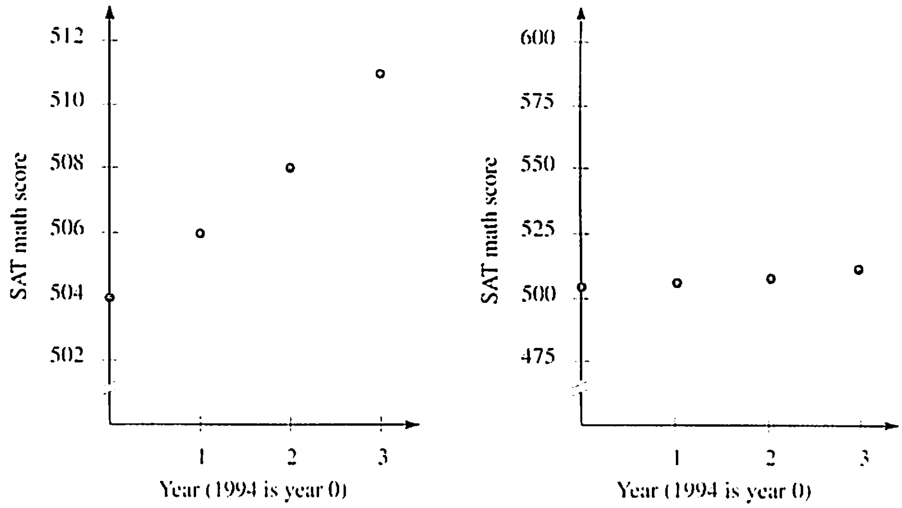
represent *years since 1965*. In other words, the year 1965 is $t = 0$, 1966 is $t = 1$, 1967 is $t = 2$, and so on. The data in the first row of Table 1 are then plotted in Figure 9(c) as the point (0, 3.345), rather than (1965, 3.345). Likewise, the second row of data in Table 1 is plotted in Figure 9(c) as (10, 4.086), and so on.

Both the bar graph and the rectangular plots in Figure 9 make it immediately clear that the world population is increasing. Is it increasing at a steady rate? Is it increasing rapidly? In fact, one needs to exercise caution in using graphs to draw conclusions about *how fast* the quantity being graphed (in this case, population) is increasing or decreasing. For instance, Figure 10 shows another graph of world population, this time covering the period 1800–1995. Figure 9(b) and Figure 10 may lead to different interpretations about the nature of world population growth. [In Figure 10, the four blue dots are the data points that appear in Figure 9(b).]

As another example about the need for caution in interpreting graphs, look at Figure 11, which shows two very different interpretations of the data for SAT

Figure 10
World population 1800–1995
Source: U.S. Census Bureau





(a) SAT math scores climbing!

(b) Little improvement shown in SAT math scores!

Figure 11
Two visualizations and interpretations of the data in Table 2

TABLE 2 SAT Math Scores: National Averages 1994–1997

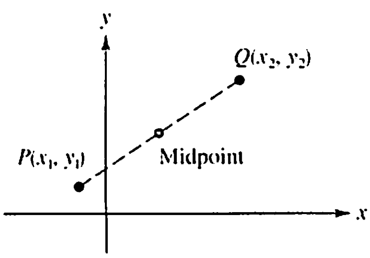
Year	SAT math score
0 (1994)	504
1 (1995)	506
2 (1996)	508
3 (1997)	511

Source: The College Board

mathematics scores in Table 2. The bottom line is that graphs are useful, even indispensible, in giving us an easy way to see general trends in data, but we must exercise care in drawing further conclusions, especially regarding rates of increase or decrease. A complete analysis of how fast a quantity is increasing or decreasing may require topics from calculus or the field of statistics. For straight-line graphs, however, the concept of *slope* (reviewed in Section 1.6) tells us definitively about rates of increase or decrease. Also, when we study functions in Chapter 3, we'll make a first step toward answering general questions about rates of change.

In Example 5 we make use of a simple result that you may recall from previous courses: the *midpoint formula*. This result is summarized in the box that follows. (For a proof of the formula, see Exercise 31 at the end of this section.)

THE MIDPOINT FORMULA



	Example
The midpoint of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	The midpoint of the line segment joining $(2, -15)$ and $(4, 5)$ is $\left(\frac{2+4}{2}, \frac{-15+5}{2}\right) = (3, -5)$

EXAMPLE 5 An application of the midpoint formula

Data concerning the amount of carbon dioxide in the atmosphere (measured in *parts per million* or *ppm*) is used by environmental scientists in the study of global warming. Table 3 provides some figures for the period 1990–1996.

TABLE 3 Atmospheric Carbon Dioxide

Year	Carbon dioxide in atmosphere (ppm)
1990	354.0
1992	356.3
1994	358.9
1996	362.6

Source: C. D. Keeling and T. P. Whorf, Scripps Institution of Oceanography

(a) Plot the data in a rectangular coordinate system. Use the variable t on the horizontal axis, with $t = 0$ corresponding to the year 1990. Use the variable c (to denote carbon dioxide levels, in ppm) on the vertical axis.
(b) Use the midpoint formula and the data for 1992 and 1994 to estimate the amount of carbon dioxide for the year 1993.
(c) Compute the *percentage error* in the estimation in part (b), given that the actual 1993 value is 357.0 ppm. The general formula for percentage error in an estimation or approximation is

$$\text{percentage error} = \frac{|(\text{actual value}) - (\text{approximate value})|}{\text{actual value}} \times 100$$

SOLUTION
(a) See Figure 12. Note that if $t = 0$ corresponds to 1990, then $t = 2$ corresponds to 1992, $t = 4$ corresponds to 1994, and $t = 6$ corresponds to 1996.

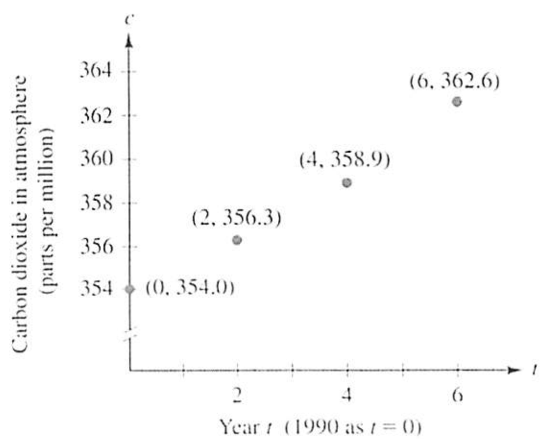


Figure 12

(b) The midpoint of the line segment joining the points (2, 356.3) and (4, 358.9) is

$$\left(\frac{2+4}{2}, \frac{356.3+358.9}{2}\right)$$

which, as you can verify, works out to

$$(3, 357.6)$$

Thus, our approximation for the carbon dioxide level in 1993 ($t = 3$) is 357.6 ppm.

(c) We have

$$\begin{aligned}\text{percentage error} &= \frac{|(\text{actual value}) - (\text{approximate value})|}{\text{actual value}} \times 100 \\ &= \frac{|357.0 - 357.6|}{357.0} \times 100 \\ &\approx 0.2\% \quad \text{using a calculator and rounding to one decimal place}\end{aligned}$$

In Figure 13 we show the given data for 1992 and 1994 along with the estimated and the actual value for 1993.

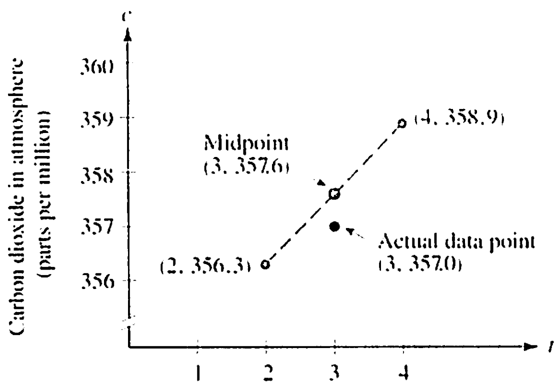


Figure 13
The midpoint of the line segment is close to the actual data point.

CAUTION Do not assume on the basis of this one example that the midpoint approximation always works as well as it does here. In this regard, be sure to work Exercise 23 at the end of this section.

EXERCISE SET 1.4

A

1. Plot the points (5, 2), (−4, 5), (−4, 0), (−1, −1), and (5, −2).

2. Draw the square $ABCD$ with vertices (corners) $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, and $D(0, -1)$.

3. (a) Draw the right triangle PQR with vertices $P(1, 0)$, $Q(5, 0)$, and $R(5, 3)$.

(b) Use the formula for the area of a triangle, $A = \frac{1}{2}bh$, to find the area of triangle PQR in part (a).

4. (a) Draw the trapezoid $ABCD$ with vertices $A(0, 0)$, $B(7, 0)$, $C(6, 4)$, and $D(4, 4)$.

(b) Compute the area of the trapezoid. (See the inside front cover of this book for the appropriate formula.)

In Exercises 5–10, calculate the distance between the given points.

5. (a) (0, 0) and (−3, 4)
(b) (2, 1) and (7, 13)

6. (a) (−1, −3) and (−5, 4)
(b) (6, −2) and (−1, 1)

7. (a) (−5, 0) and (5, 0)
(b) (0, −8) and (0, 1)

8. (a) (−5, −3) and (−9, −6)
(b) $(\frac{9}{2}, 3)$ and $(-2\frac{1}{2}, -1)$

9. $(1, \sqrt{3})$ and $(-1, -\sqrt{3})$

10. (−3, 1) and (374, −335)

11. Which point is farther from the origin?
(a) (3, −2) or $(4, \frac{1}{2})$
(b) (−6, 7) or (9, 0)

12. Use the distance formula to show that, in each case, the triangle with given vertices is an isosceles triangle.
(a) $(0, 2), (7, 4), (2, -5)$
(b) $(-1, -8), (0, -1), (-4, -4)$
(c) $(-7, 4), (-3, 10), (1, 3)$

13. In each case, determine whether the triangle with the given vertices is a right triangle.
(a) $(7, -1), (-3, 5), (-12, -10)$
(b) $(4, 5), (-3, 9), (1, 3)$
(c) $(-8, -2), (1, -1), (10, 19)$

14. (a) Two of the three triangles specified in Exercise 13 are right triangles. Find their areas.
(b) Calculate the area of the remaining triangle in Exercise 13 by using the following formula for the area A of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3):

$$A = \tfrac{1}{2}|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3|$$

The derivation of this formula is given in Exercise 34.
(c) Use the formula given in part (b) to check your answers in part (a).

15. Use the formula given in Exercise 14(b) to calculate the area of the triangle with vertices $(1, -4)$, $(5, 3)$, and $(13, 17)$. What do you conclude?

16. The coordinates of points A, B, and C are $A(-4, 6)$, $B(-1, 2)$, and $C(2, -2)$.
(a) Show that $AB = BC$ by using the distance formula.
(b) Show that $AB + BC = AC$ by using the distance formula.
(c) What can you conclude from parts (a) and (b)?

In Exercises 17 and 18, find the midpoint of the line segment joining points P and Q.

17. (a) $P(3, 2)$ and $Q(9, 8)$
(b) $P(-4, 0)$ and $Q(5, -3)$
(c) $P(3, -6)$ and $Q(-1, -2)$

18. (a) $P(12, 0)$ and $Q(12, 8)$
(b) $P(\frac{3}{5}, -\frac{2}{3})$ and $Q(0, 0)$
(c) $P(1, \pi)$ and $Q(3, 3\pi)$

In Exercises 19 and 20, the given points P and Q are the endpoints of a diameter of a circle. Find **(a)** *the center of the circle;* **(b)** *the radius of the circle.*

19. $P(-4, -2)$ and $Q(6, 4)$

20. $P(1, -3)$ and $Q(-5, -5)$

21. (a) Using a coordinate system similar to the one shown in the following figure (or a photocopy), plot the two points from Table A corresponding to the data for the years 1984 and 1992. Ignore the information about the name of the network.
(b) Use the midpoint formula with the two points that you plotted in part (a) to obtain an approximation for the dollar amount paid per TV hour for 1988. (Round the answer to one decimal place.)
(c) Compute the percentage error in the approximation in part (b). The actual 1988 value is given in the table.

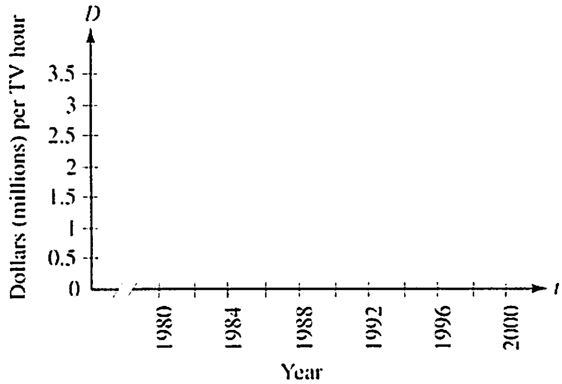


TABLE A How Much the Networks Paid (per TV hour) to Televise the Winter Olympics, 1980–1998

Year (network)	1980 (ABC)	1984 (ABC)	1988 (ABC)	1992 (CBS)	1994 (CBS)	1998 (CBS)
Millions of dollars per TV hour	0.3	1.5	3.3	2.1	2.5	2.9

Source: World Features Syndicate

22. (a) Set up a coordinate system with the horizontal t-axis (running from 0 to at least 6) representing years af-ter 1990 and the vertical P-axis (running from 14 to at least 27) representing percentage of sales due to imports; then use it to plot the data in Table B below. *Note:* you should use a broken vertical axis as in Figures 11 through 13.
(b) What are the coordinates of the point in your graph that corresponds to the data for 1990? For 1992?
(c) Use the midpoint formula with the two points that you listed in part (b) to estimate the percentage of import sales for the year 1991. (Round the answer to one decimal place.)
(d) Compute the percentage error in the estimate in part (c), given that the actual 1991 value is 24.9%. [The formula for percentage error is given in Example 5(c).]

TABLE B Percentage of Retail New Car Sales in United States due to Imports, 1990–1996

Year	1990	1992	1994	1996
Percentage due to imports	25.8	23.6	19.3	14.9

Source: American Automobile Manufacturers Assn.

23. Over the past two decades the Internet has grown very rapidly. Figures A and B provide estimates for the number

n of Internet host computers, worldwide, for the years t = 1995–1997 and 1985–1987. Source: *Network Wizards* (http//www.nw.com)

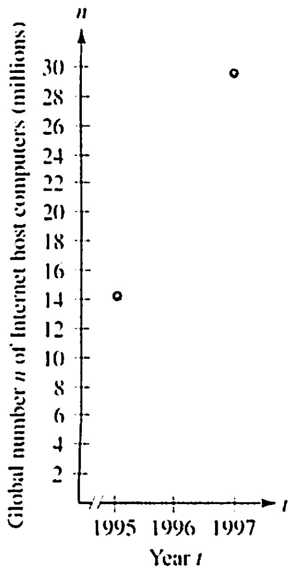


Figure A

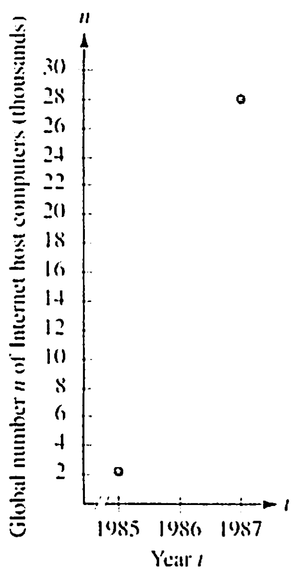


Figure B

(a) Use Figure A to complete the following table. Round the values of n to the nearest two million. Then use the midpoint formula and the numbers in your table to estimate the global number of Internet host computers for the year 1996.

t	1995	1997
n		

(b) Use Figure B to complete the following table. Round the values of n to the nearest two thousand. Then use the midpoint formula and the numbers in your table to estimate the global number of Internet host computers for the year 1986.

t	1985	1987
n		

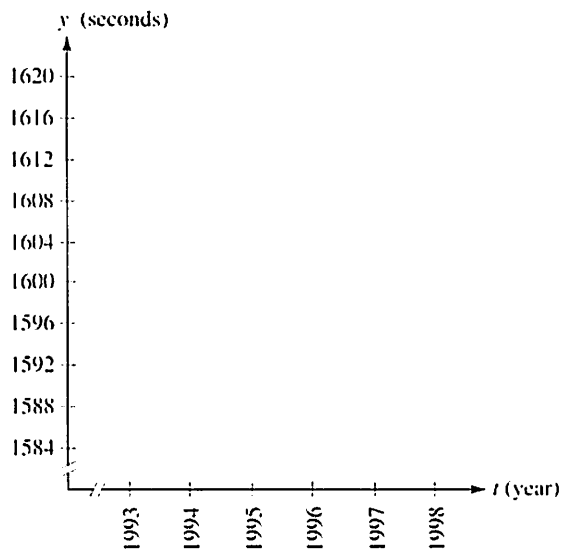
(c) Compute the percentage errors to determine which estimate, the one for 1986 or the one for 1996, is more accurate. Use the following data from *Network Wizards* (http//www.nw.com) in computing the percentage errors: The number of host computers for 1986 and 1996 were 5089 and 21,819,000, respectively. (Round each answer to the nearest one percent.)

Note: You'll find out in part (c) that one estimate is very good, the other is way off. The point here is that without more initial information, it's hard to say whether the midpoint formula will produce a useful estimate. In subsequent chapters, we'll use *functions* and larger data sets to obtain more reliable estimates.

24. Have you or a friend ever run in a 10K (10,000 meter) race? When the author polled his precalculus class at UCLA in Fall 1997, he found that there were five students in the class (of 160) who said they had run a 10K in under 50 minutes. Of those five, two (one male, one female) said they had run a 10K in under 40 minutes. The world record for this event is well under 30 minutes. In this exercise you'll look at some of the world records in this event over the past decade.

(a) The table that follows lists the world records in the (men's) 10,000 meter race as of the end of the years 1993, 1995, and 1997. After converting the times into seconds, plot the three points corresponding to these records in a coordinate system similar to the one shown.

Year	Time	Runner
1993	26:58.38	Yobes Ondieki (Kenya)
1995	26:43.53	Haile Gebrselassie (Kenya)
1997	26:27.85	Paul Tergat (Kenya)



(b) Use the midpoint formula and the data for 1993 and 1995 to compute an estimate for what the world record might have been by the end of 1994. Then compute the percentage error (rounded to two decimal places), given that the record at the end of 1994 was 26:52.23 (set by William Seigei of Kenya). Was your estimate too high or too low?

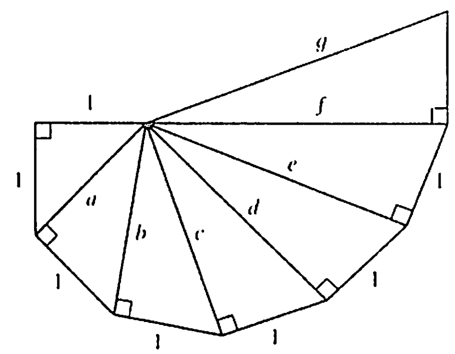
(c) Use the midpoint formula and the data for 1995 and 1997 to compute an estimate for what the world record might have been by the end of 1996. Then compute the percentage error given that the record at the end of 1996 was 26:38.08 (set by Salah Hissou of Morocco). Was your estimate too high or too low? Is the percentage error more or less than that obtained in part (b)?

(d) Using a coordinate system similar to the one shown in part (a), or using a photocopy, plot the points corresponding to the (actual, not estimated) world records for the years 1993, 1994, 1995, 1996, 1997, and 1998. Except for 1998, all the records have been given above. The world record at the end of 1998 was 26:22.75 (set by Haile Gebrselassie of Kenya). Use the picture you obtain to say whether or not the record times seem to be leveling off.

25. **(a)** Sketch the parallelogram with vertices $A(-7, -1)$, $B(4, 3)$, $C(7, 8)$, and $D(-4, 4)$.
(b) Compute the midpoints of the diagonals $\overline{AC}$ and $\overline{BD}$.
(c) What conclusion can you draw from part (b)?

26. The vertices of $\triangle ABC$ are $A(1, 1)$, $B(9, 3)$, and $C(3, 5)$.
(a) Find the perimeter of $\triangle ABC$.
(b) Find the perimeter of the triangle that is formed by joining the midpoints of the three sides of $\triangle ABC$.
(c) Compute the ratio of the perimeter in part (a) to the perimeter in part (b).
(d) What theorem from geometry provides the answer for part (c) without using the results in (a) and (b)?

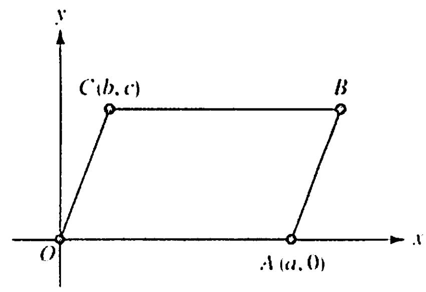
27. Use the Pythagorean theorem to find the length a in the figure. Then find b, c, d, e, f, and g.



Note: This figure provides a geometric construction for the irrational numbers $\sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}$, where n is a nonsquare natural number. According to Boyer's *A History of Mathematics*, 2nd ed. (New York: John Wiley & Sons, Inc., 1991), "Plato . . . says that his teacher Theodorus of Cyrene . . . was the first to prove the irrationality of the square roots of the nonsquare integers from 3 to 17, inclusive. It is not known how he did this, nor why he stopped with $\sqrt{17}$." One plausible reason for Theodorus's stopping with $\sqrt{17}$ may have to do with the figure shown here. Theodorus may have known that the figure begins to overlap itself at the stage where $\sqrt{18}$ would be constructed.

28. (A numerologist's delight) Using the Pythagorean theorem and your calculator, compute the area of a right triangle in which the lengths of the hypotenuse and one leg are 2045 and 693, respectively.

29. The diagonals of a parallelogram bisect each other. Steps (a), (b), and (c) outline a proof of this theorem. (See Exercise 25 for a particular instance of this theorem.)
(a) In the parallelogram $OABC$ shown in the figure, check that the coordinates of B must be $(a + b, c)$.
(b) Use the midpoint formula to calculate the midpoints of diagonals $\overline{OB}$ and $\overline{AC}$.
(c) The two answers in part (b) are identical. This shows that the two diagonals do indeed bisect each other, as we wished to prove.



30. Prove that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the four sides. (Use the figure in Exercise 29.)

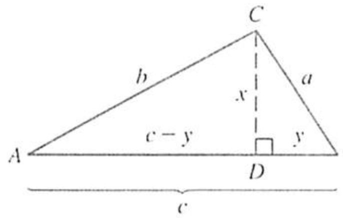
31. Suppose that the coordinates of points P, Q, and M are

$$P(x_1, y_1) \qquad Q(x_2, y_2)$$
$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

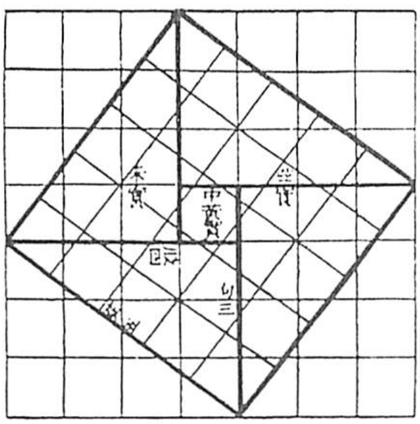
Follow steps (a) and (b) to prove that M is the midpoint of the line segment from P to Q.
(a) By computing both of the distances PM and MQ, show that $PM = MQ$. (This shows that M lies on the perpendicular bisector of line segment $\overline{PQ}$, but it does not show that M actually lies on $\overline{PQ}$.)
(b) Show that $PM + MQ = PQ$. (This shows that M does lie on $\overline{PQ}$.)

32. This problem outlines one of the shortest proofs of the Pythagorean theorem. The proof was discovered by the Hindu mathematician Bhāskara (1114–ca. 1185). (For other proofs, see the next exercise and also Exercise 100 on page 83.) In the figure we are given a right triangle ACB with the right angle at C, and we want to prove that $a^2 + b^2 = c^2$. In the figure, $\overline{CD}$ is drawn perpendicular to $\overline{AB}$.
(a) Check that $\angle CAD = \angle DCB$ and that $\triangle BCD$ and $\triangle BAC$ are similar.
(b) Use the result in part (a) to obtain the equation $a/y = c/a$, and conclude that $a^2 = cy$.
(c) Show that $\triangle ACD$ is similar to $\triangle ABC$, and use this to deduce that $b^2 = c^2 - cy$.

(d) Combine the two equations deduced in parts (b) and (c) to arrive at $a^2 + b^2 = c^2$.



33. One of the oldest and simplest proofs of the Pythagorean theorem is found in the ancient Chinese text *Chou Pei Suan Ching*. This text was written during the Han period (206 B.C.–A.D. 222), but portions of it may date back to 600 B.C. The proof in *Chou Pei Suan Ching* is based on this diagram from the text. In this exercise we explain the details of the proof.



A diagram accompanying a proof of the "Pythagorean" theorem in the ancient Chinese text *Chou Pei Suan Ching* [from *Science and Civilisation in China*, vol. 3, by Joseph Needham (Cambridge, England: Cambridge University Press, 1959)].

(a) Starting with the right triangle in Figure A, we make four replicas of this triangle and arrange them to form the pattern shown in Figure B. Explain why the outer quadrilateral in Figure B is a square.

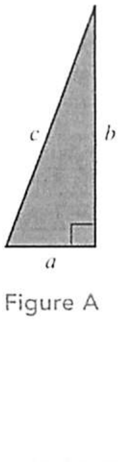


Figure A

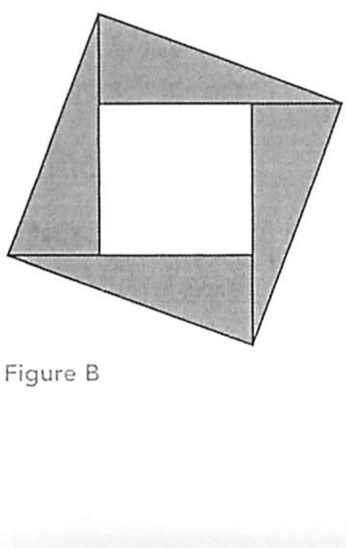


Figure B

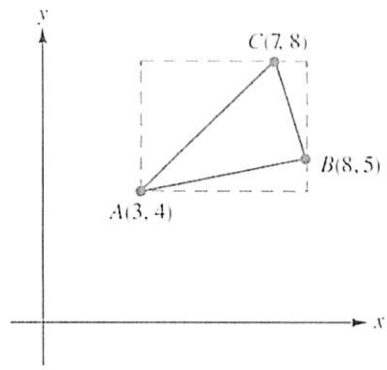
(b) The unshaded region in the center of Figure B is a square. What is the length of each side?

(c) The area of the outer square in Figure B is $(\text{side})^2 = c^2$. This area can also be computed by adding up the areas of the four right triangles and the inner square. Compute the area in this fashion. After simplifying, you should obtain $a^2 + b^2$. Now conclude that $a^2 + b^2 = c^2$, since both expressions represent the same area.

C

34. This problem indicates a method for calculating the area of a triangle when the coordinates of the three vertices are given.

(a) Calculate the area of $\triangle ABC$ in the figure.



Hint: First calculate the area of the rectangle enclosing $\triangle ABC$, and then subtract the areas of the three right triangles.

(b) Calculate the area of the triangle with vertices (1, 3), (4, 1), and (10, 4). *Hint:* Work with an enclosing rectangle and three right triangles, as in part (a).

(c) Using the same technique that you used in parts (a) and (b), show that the area of the triangle in the following figure is given by

$$A = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

Remark: If we use absolute value signs instead of the parentheses, then the formula will hold regardless of the relative positions or quadrants of the three vertices. Thus the area of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$A = \frac{1}{2}|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3|$$